

The exterior field of slowly and rapidly rotating neutron stars: Rehabilitating spacetime metrics involving hyperextreme objects

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The 4-parameter exact solution presumably describing the exterior gravitational field of a generic neutron star is presented in a concise explicit form defined by only three potentials. In the equatorial plane, the metric functions of the solution are found to be given by particularly simple expressions that make them very suitable for the use in concrete applications. Following Pappas and Apostolatos, we perform a comparison of the multipole structure of the solution with the multipole moments of the known physically realistic Berti-Stergioulas numerical models of neutron stars to argue that the hyperextreme sectors of the solution are not less (but possibly even more) important for the correct description of rapidly rotating neutron stars than the subextreme sector involving exclusively the black-hole constituents. We have also worked out in explicit form an exact analog of the well-known Hartle-Thorne approximate metric.

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I. INTRODUCTION

In 1995 a general equatorially symmetric two-soliton solution of the Einstein-Maxwell equations [1] (henceforth referred to as the MMR solution) was constructed as a physically and astrophysically important application of analytic formulas determining the extended N -soliton metric [2] obtained with the aid of Sibgatullin's integral method [3]. Since then, various particular cases of that solution were analyzed in the literature in relation with the exterior field of neutron stars (NSs). Thus, for instance, as an alternative to the well-known approximate Hartle-Thorne metric [4], in a series of papers [5] Sibgatullin and Sunyaev used for the description of the NS exterior geometry a 3-parameter vacuum specialization of the MMR metric [6], and they demonstrated that the exact solution with arbitrary parameters of mass, angular momentum and mass-quadrupole moment was in good agreement with the numerical models and data from the well-known paper of Cook, Shapiro and Teukolsky [7] obtained for various equations of state (EOS). They also discovered some universal (independent of the EOS) properties of neutron stars with regard to the rescaled dimensionless multipole moments. A limiting case of the MMR solution was considered in the paper [8], and several authors then studied its 3-parameter vacuum subcase containing, similar to the solution analyzed by Sibgatullin and Sunyaev, an arbitrary mass-quadrupole parameter [9–12]. A comparison of the latter subclass with the numerical models of NSs, performed by Berti and Stergioulas [10] with the help of an advanced numerical code [13, 14], revealed in particular that the 3-parameter analytic solution was better suited for modeling the geometry around rapidly rotating NSs, giving at the same time a worse matching with the numerical data for slowly rotating NSs. Even though afterwards the papers of Pappas and Apostolatos [15, 16] helped to considerably reduce the discrepancies between the analytical and numerical models, still the paper [10] highlighted the desirability of an additional arbitrary parameter (representing a rotational octupole moment) in the analytic solutions that pretend to describe a generic NS. The desired octupole parameter is naturally contained in the vacuum sector of the MMR metric defined, within the framework of the Ernst formalism [17, 18] and Sibgatullin's method [3], by the axis data of the form [1]¹

$$\mathcal{E}(\rho = 0, z) \equiv e(z) = \frac{(z - m - ia)(z + ib) + k}{(z + m - ia)(z + ib) + k}, \quad (1)$$

where the four arbitrary real parameters m , a , k and b are associated, respectively, with the mass, angular momentum, mass-quadrupole and angular-momentum-octupole moments of the source. In recent years, to a large extent due to the efforts of Pappas and Apostolatos [19–21], the corresponding vacuum metric was shown to be the best analytical

¹ Note that, compared to the original paper [1], we have introduced a formal sign change $k \rightarrow -k$ in the axis data (1).

approximation to the exterior gravitational field of any kind of unmagnetized rotating NSs, and lately such an assessment has been strongly supported by the discovery of various universal properties and relations for NSs [22, 23], accompanied by the explanation of the physical mechanisms that may lie behind that universal behavior [24]. Nonetheless, despite all its notable properties, the vacuum MMR solution defined by (1) has not yet found a widespread use among the researchers, and in our opinion this might partly be attributed to a very complicated form that was given to it in the paper [20] where Pappas and Apostolatos apparently did not use the most rational way of the calculation of the metric function ω allowed by Sibgatullin's method. Therefore, one of the main objectives of the present paper will be giving a concise form for the whole MMR 4-parameter vacuum solution, together with the remarkably simple expressions of its metric functions in the equatorial plane, what we believe must make this solution very suitable for direct use in astrophysical applications even by non-experts in the solution generating techniques. Moreover, since it was already formally shown in the paper [20] how different branches of the two-soliton solution are related to the numerical results of Berti and Stergioulas [10], in the present paper we are going to reconsider this question in more detail, demonstrating in particular how the concrete type of the solution (subextreme, hyperextreme or mixed) can be read off from the numerical data of the paper [10] directly at the level of the multipole moments, and we will speculate about a possible impact our analysis might have on raising the physical status of hyperextreme spacetimes.

Our paper is organized as follows. A concise form of the MMR vacuum solution will be given in the next section, together with our clarifying remarks on finding the metric function ω in the vacuum and electrovacuum cases by means of Sibgatullin's method; here we also obtain a simple representation of the solution in the equatorial plane. In Sec. III we first consider a reparametrization of the axis data (1) in terms of four multipole moments and then show how this permits one to study the involvement of the subextreme and hyperextreme constituents in our two-soliton solution when the numerical results of Berti and Stergioulas [10] are being used as matching data. Furthermore, to make the interior structure of the MMR vacuum solution more comprehensible to the reader, we will illustrate it by comparing the latter solution with the extended two-soliton solution from our paper [25]. Here we also consider an exact analog of the Hartle-Thorne approximate metric. Sec. IV is devoted to the discussion of possible physical implications of the results obtained and to concluding remarks.

II. THE MMR 4-PARAMETER VACUUM SOLUTION

Unlike the paper [20], where the expression (1) was presented by Pappas and Apostolatos as just an *ansatz*, we first of all would like to remark that Eq. (1) is not an intuitive result or some fortunate guessing, but in reality it is a direct outcome of thorough analysis of the multipole structure of the extended two-soliton electrovac solution performed in the paper [26]. The four arbitrary parameters in (1) correspond to four arbitrary Geroch-Hansen relativistic multipole moments [27, 28], and the form of these multipoles in terms of the parameters m , a , b and k can be easily found from (1) with the aid of the Fodor-Hoenselaers-Perjés procedure [29], yielding [1]

$$M_0 = m, \quad M_2 = -m(a^2 + k), \quad J_1 = ma, \quad J_3 = -m[a^3 + k(2a - b)]. \quad (2)$$

Here M_0 stands for the total mass, M_2 is the mass-quadrupole moment, whereas J_1 is the total angular momentum of the source and J_3 its angular-momentum-octupole moment. It is then clear that the parameters m and a have the same physical interpretation as in the Kerr solution [30], so that the two remaining parameters b and k actually describe the deviations of the MMR vacuum solution from the Kerr spacetime.

Although the expression of the complex potential $\mathcal{E}(\rho, z)$ defined by (1) and the corresponding metric functions $f(\rho, z)$, $\gamma(\rho, z)$ and $\omega(\rho, z)$ entering the Weyl-Papapetrou stationary axisymmetric line element

$$ds^2 = f^{-1}[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\varphi^2] - f(dt - \omega d\varphi)^2, \quad (3)$$

are readily obtainable from the 6-parameter electrovac solution of the paper [1] by just setting in it to zero the charge parameter q and the magnetic dipole parameter c ,² Pappas and Apostolatos still opted in [20] for their own rederivation of the vacuum solitonic solution from the general formulas of the paper [2]. However, they were seemingly unaware of the important fact that in the pure vacuum case, when the electromagnetic field is absent, the knowledge of only one of the potentials G or H is really needed for the construction of the metric coefficient ω , not both of them. Actually, for their purpose Pappas and Apostolatos should have better used the formulas (2.1) of the paper [25] which already take into account the peculiarities of the vacuum case. Having in mind the idea of improving the presentation

² Note that the expression of the function A in [1] contained some misprints that were later rectified in [31].

of the vacuum MMR metric, recently we have carefully revised our earlier work on the extended two-soliton solutions, exploring in particular various ways of writing the metric function ω of which we have finally chosen the one that looked to us more attractive than the others. However, before the presentation of the metric functions of the MMR solution, below we first write down the form of the Ernst potential \mathcal{E} of the latter solution [1, 31]:

$$\begin{aligned}
\mathcal{E} &= (A - B)/(A + B), \\
A &= \kappa_+^2 \{ [m^2(d - ab + 2b^2) - (a - b)^2(d - ab - k)](R_+r_- + R_-r_+) \\
&\quad - i\kappa_- [(a - b)(d - ab - k) + m^2b](R_+r_- - R_-r_+) \} \\
&\quad + \kappa_-^2 \{ [m^2(d + ab - 2b^2) - (a - b)^2(d + ab + k)](R_+r_+ + R_-r_-) \\
&\quad - i\kappa_+ [(a - b)(d + ab + k) - m^2b](R_+r_+ - R_-r_-) \} - 4m^2kd(R_+R_- + r_+r_-), \\
B &= m\kappa_+\kappa_- \{ d[\kappa_+\kappa_-(R_+ + R_- + r_+ + r_-) - (m^2 - a^2 + b^2)(R_+ + R_- - r_+ - r_-)] \\
&\quad + ibd[(\kappa_+ + \kappa_-)(R_+ - R_-) + (\kappa_+ - \kappa_-)(r_- - r_+)] \\
&\quad + i[b(m^2 - a^2) - ak][(\kappa_+ + \kappa_-)(r_+ - r_-) + (\kappa_+ - \kappa_-)(R_- - R_+)] \},
\end{aligned} \tag{4}$$

where

$$\begin{aligned}
R_{\pm} &= \sqrt{\rho^2 + \left(z \pm \frac{1}{2}(\kappa_+ + \kappa_-)\right)^2}, \quad r_{\pm} = \sqrt{\rho^2 + \left(z \pm \frac{1}{2}(\kappa_+ - \kappa_-)\right)^2}, \\
\kappa_{\pm} &= \sqrt{m^2 - a^2 - b^2 - 2k \pm 2d}, \quad d = \sqrt{(ab + k)^2 - m^2b^2}.
\end{aligned} \tag{5}$$

It is not difficult to verify that on the upper part of the symmetry axis ($\rho = 0$, $z > \text{Re}[(\kappa_+ + \kappa_-)/2]$) the above potential \mathcal{E} takes the form (1). Of course, it can also be readily checked with a computer analytical program that \mathcal{E} defined by (4) and (5) satisfies identically the Ernst equation

$$\text{Re}(\mathcal{E})(\mathcal{E}_{,\rho,\rho} + \rho^{-1}\mathcal{E}_{,\rho} + \mathcal{E}_{,z,z}) = \mathcal{E}_{,\rho}^2 + \mathcal{E}_{,z}^2 \tag{6}$$

(the comma in subindices denotes partial differentiation).

While the corresponding metric functions f and γ of the vacuum MMR solution can be written in terms of the above potentials A and B only, the form of the remaining function ω involves the additional potential G which we calculated with the aid of the formulas of our paper [25] in a bit more rational way than this was done in the paper [1]; the final expressions for f , γ and ω are the following:

$$\begin{aligned}
f &= \frac{A\bar{A} - B\bar{B}}{(A + B)(\bar{A} + \bar{B})}, \quad e^{2\gamma} = \frac{A\bar{A} - B\bar{B}}{16d^2\kappa_+^4\kappa_-^4R_+R_-r_+r_-}, \quad \omega = 2(a - b) - \frac{2\text{Im}[G(\bar{A} + \bar{B})]}{A\bar{A} - B\bar{B}}, \\
G &= -zB + \kappa_+^2\kappa_-^2[m^2(d - ab + 2b^2) - (a - b)^2(d - ab - k)](R_-r_+ - R_+r_-) \\
&\quad + \kappa_+\kappa_-^2[m^2(d + ab - 2b^2) - (a - b)^2(d + ab + k)](R_-r_- - R_+r_+) + i\kappa_+^2\kappa_-^2 \\
&\quad \times \{ d(a - b)(R_+ + R_-)(r_+ + r_-) + [(a - b)(ab + k) - m^2b](R_+ - R_-)(r_+ - r_-) \\
&\quad + mbd(R_+ + R_- + r_- + r_+) \} + md\kappa_+\kappa_- \{ \kappa_+(d + b^2 + k)(R_- - R_+ - r_- + r_+) \\
&\quad + \kappa_-(d - b^2 - k)(R_+ - R_- - r_- + r_+) + i[(a - b)(ab + b^2 + 2k) - m^2b] \\
&\quad \times (R_+ + R_- - r_- - r_+) \}
\end{aligned} \tag{7}$$

(a bar over a symbol means complex conjugation), and apparently our way of expressing the metric function ω is by far simpler than the formulas (14) and (B14)-(B20) of [20].

It is worth noting that the function ω of the MMR vacuum solution that follows straightforwardly from the formulas (3.11) of [1] (by setting $q = c = 0$ and changing k to $-k$) also has quite a reasonable form somehow overlooked by Pappas and Apostolatos, namely,

$$\begin{aligned}
\omega &= -\frac{2\text{Im}[m\bar{L}(A + B)]}{A\bar{A} - B\bar{B}}, \\
L &= \kappa_+^2\kappa_-^2[d(z - ia)(R_+ + R_- + r_+ + r_-) + ibm(R_+ - R_-)(r_+ - r_-)] \\
&\quad - imk(a - b)[4d(R_+R_- + r_+r_-) - \kappa_+^2(R_+r_- + R_-r_+) + \kappa_-^2(R_+r_+ + R_-r_-)] \\
&\quad + \kappa_+\kappa_- \{ m[\kappa_+(d + b^2)(R_+r_- - R_-r_+) + \kappa_-(d - b^2)(R_+r_+ - R_-r_-)] \\
&\quad + d[(z - ia)(a^2 - b^2 - m^2) - 2ik(a - b)](R_+ + R_- - r_+ - r_-) \\
&\quad + d(ab + k + ibz)[(\kappa_+ + \kappa_-)(R_+ - R_-) + (\kappa_+ - \kappa_-)(r_- - r_+)] \\
&\quad + [k^2 + abk + (a + iz)(a^2b + ak - bm^2)][(\kappa_+ + \kappa_-)(r_- - r_+) \\
&\quad + (\kappa_+ - \kappa_-)(R_+ - R_-)] \},
\end{aligned} \tag{8}$$

where the factor 2 in the above expression of ω reflects the fact that the contributions of the terms involving the potentials E and L in the formula (3.11) of [1] for ω are identical in the absence of electromagnetic field, so that only one of these potentials is needed in such case for the construction of the field ω . Nonetheless, we ourselves still incline to the form (7) for ω because, on the one hand, it is slightly more concise than the expression (8) and, on the other hand, we used it for elaborating a nice representation of the MMR vacuum metric in the equatorial plane that will be considered below.

Since in many practical astrophysical applications of the NS models (the geodesic motion of test particles, the existence of innermost stable circular orbits, energy release on the surface of a NS by accreting matter, etc.) the analysis is usually restricted to the equatorial plane of the exterior field, it is desirable to have compact analytical expressions of all metric coefficients for this important special domain defined in cylindrical coordinates as $z = 0$, $\rho \geq 0$. Let us note that a very concise “equatorial” form for the 3-parameter solution discussed in the papers [5] was found by Sibgatullin and Sunyaev (see Eqs. (26) in the first paper of Ref. [5]), and its knowledge was sufficient for being able to study extensively various physical properties of NSs. Though the additional parameter b in the vacuum MMR 4-parameter solution apparently complicates the general form of the metrical fields compared to its particular 3-parameter specialization considered by Sibgatullin and Sunyaev, we notwithstanding have been able to obtain an analogous very simple representation for our more general solution in the equatorial plane, and this required, apart from just setting $z = 0$ in the formulas (4), (5) and (7), some additional algebraic manipulations that eventually led us to the following elegant result:

$$\begin{aligned} f &= \frac{A-B}{A+B}, \quad e^{2\gamma} = \frac{A^2 - B^2}{(r_+ + r_-)^4 r_+^2 r_-^2}, \quad \omega = -\frac{2mW}{A-B}, \\ A &= (r_+ + r_-)^2 r_+ r_- - m^2 k, \\ B &= m(r_+ + r_-)(r_+ r_- + \rho^2 - b^2 - k), \\ W &= (r_+ + r_-)[a(r_+ r_- + \rho^2 - b^2) - bk] + mk(a - b), \\ r_{\pm} &= \sqrt{\rho^2 + \frac{1}{4}(\kappa_+ \pm \kappa_-)^2}. \end{aligned} \quad (9)$$

The above formulas can also be rewritten in dimensionless form by introducing

$$j = \frac{a}{m}, \quad \beta = \frac{b}{m}, \quad \kappa = \frac{k}{m^2}, \quad r = \frac{\rho^2}{m^2}, \quad r_{\pm} = \frac{r_{\pm}}{m}, \quad (10)$$

and thus yielding

$$\begin{aligned} f &= \frac{\mathcal{A} - \mathcal{B}}{\mathcal{A} + \mathcal{B}}, \quad e^{2\gamma} = \frac{\mathcal{A}^2 - \mathcal{B}^2}{(r_+ + r_-)^4 r_+^2 r_-^2}, \quad \frac{\omega}{m} = -\frac{2\mathcal{W}}{\mathcal{A} - \mathcal{B}}, \\ \mathcal{A} &= (r_+ + r_-)^2 r_+ r_- - \kappa, \\ \mathcal{B} &= (r_+ + r_-)(r_+ r_- + r - \beta^2 - \kappa), \\ \mathcal{W} &= (r_+ + r_-)[j(r_+ r_- + r - \beta^2) - \beta\kappa] + \kappa(j - \beta), \end{aligned} \quad (11)$$

with r_{\pm} having the explicit form

$$\begin{aligned} r_{\pm} &= \sqrt{r + \frac{1}{4} \left(\sqrt{1 - j^2 - \beta^2 - 2\kappa + 2\delta} \pm \sqrt{1 - j^2 - \beta^2 - 2\kappa - 2\delta} \right)^2}, \\ \delta &\equiv \sqrt{(\kappa + j\beta)^2 - \beta^2}. \end{aligned} \quad (12)$$

The expressions obtained by Sibgatullin and Sunyaev for the 3-parameter quadrupole solution in the equatorial plane are recovered from (11) and (12) by putting $\beta = 0$, $\delta = \kappa$ and remembering that our κ is their b .

Therefore, we have obtained the desired representation of the vacuum MMR solution and its form in the equatorial plane. We now turn to the discussion of the multipole structure of this solution in relation with the type of the objects that may formally be identified as its sources.

III. RELATIONSHIPS BETWEEN THE MMR SOLUTION, EDK SOLUTION AND NUMERICAL SOLUTIONS OF BERTI AND STERGIOULAS

The main advantage of the extended multi-soliton solutions [2, 25] that were constructed by means of Sibgatullin’s method, and to which belongs in particular the MMR solution, is that the parameters they contain correspond to

arbitrary multipole moments, so that in the pure vacuum case which is of interest to us in this paper they naturally describe in a unified manner arbitrary combinations of the subextreme and hyperextreme Kerr-NUT constituents [32]. As it follows from (2), the four parameters in the axis data (1) do correspond to four multipole moments, which means that (1) is reparametrizable in terms of the multipole moments alone. This can be done either by inverting formulas (2), namely,

$$m = M_0, \quad a = \frac{J_1}{M_0}, \quad k = -\frac{M_2}{M_0} - \frac{J_1^2}{M_0^2}, \quad b = \frac{-M_0 J_3 + M_2 J_1}{M_0 M_2 + J_1^2} + \frac{J_1}{M_0}, \quad (13)$$

with the subsequent substitution of (13) into (1), or by using directly formula (2.8) of [1] in which one has to take into account that in the equatorially symmetric case the complex quantities m_n , $n = \overline{1, 4}$, are related to M_n and J_n by

$$m_0 = M_0, \quad m_1 = iJ_1, \quad m_2 = M_2, \quad m_3 = iJ_3. \quad (14)$$

However, since the problem of parametrizing the axis data of the general multi-soliton vacuum solution in terms of the multipole moments was solved in our paper [25], below we shall simply use the determinantal expressions obtained there and apply them to the $N = 2$ case, with m_n defined by (14). Then for $e(z)$ we get the following representation:

$$e(z) = \frac{e_-(z)}{e_+(z)}, \quad e_{\pm}(z) = (L_2)^{-1} \begin{vmatrix} z^2 \pm M_0 z \pm iJ_1 & M_2 & iJ_3 \\ z \pm M_0 & iJ_1 & M_2 \\ 1 & M_0 & iJ_1 \end{vmatrix}, \quad L_2 = \begin{vmatrix} iJ_1 & M_2 \\ M_0 & iJ_1 \end{vmatrix}, \quad (15)$$

where $e_+(z)$ and $e_-(z)$ stand for $R(z)$ and $P(z)$ of the paper [25], respectively. Note that the ratio of two quadratic polynomials in z determining $e(z)$ in (15) degenerates to the ratio of two linear in z polynomials when $L_2 \equiv -M_0 M_2 - J_1^2 = 0$, $M_0 \neq 0$, in which case the resulting $e(z)$ just defines the Kerr solution.

We emphasize that the form (15) of $e(z)$ is fully equivalent to the initial axis data (1) of the vacuum MMR solution, and therefore (15) can be used to analyze the structure of this solution on a par with (1). The main benefit of having the representation (15) of $e(z)$ in terms of the multipole moments consists in the possibility to use it for establishing a straightforward correspondence between the MMR solution and the numerical solutions of Berti and Stergioulas [10], and in particular to answer an important question of which types (subextreme or hyperextreme) of the Kerr-NUT sources constituting the MMR solution correspond to each numerical Berti-Stergioulas model defined by a concrete numerical set of multipoles $\{M_0, J_1, M_2, J_3\}$. Recall that in Sibgatullin's method, when the consideration is restricted to the pure vacuum case, the roots α_n of the equation

$$e(z) + \bar{e}(z) = 0 \quad (16)$$

play a fundamental role as they enter in the ultimate expressions of the metric functions, in this way introducing the parameters of the initial axis data into the final formulas. For example, the four roots of equation (16) with the MMR data (1) are contained in the expressions (5) for the functions R_{\pm} and r_{\pm} as the constant quantities

$$\alpha_1 = -\alpha_4 = \frac{1}{2}(\kappa_+ + \kappa_-), \quad \alpha_2 = -\alpha_3 = \frac{1}{2}(\kappa_+ - \kappa_-), \quad (17)$$

and it follows from the mathematical structure of (16), in the case of rational axis data leading to polynomial equations with real coefficients, that the roots α_n can take either real values or occur in complex conjugate pairs. Then a pair of real α 's in the MMR vacuum solution defines a subextreme Kerr-NUT constituent, while a pair of complex conjugate α 's determines a hyperextreme Kerr-NUT constituent. In application to the reparametrized axis data (15) this means that the substitution of (15) into (16), yielding the biquadratic equation

$$\begin{aligned} z^4 + \left(M_0^2 + \frac{M_2^2 + 2J_1 J_3}{M_0 M_2 + J_1^2} + \frac{M_0(M_2^3 + 2M_2 J_1 J_3 - M_0 J_3^2)}{(M_0 M_2 + J_1^2)^2} \right) z^2 \\ + \left(\frac{M_2^2 + J_1 J_3}{M_0 M_2 + J_1^2} \right)^2 - \left(J_1 + \frac{M_0(M_2 J_1 - M_0 J_3)}{M_0 M_2 + J_1^2} \right)^2 = 0, \end{aligned} \quad (18)$$

permits us, by solving (18) for any given set of multipoles $\{M_0, J_1, M_2, J_3\}$ from [10] determining a particular Berti-Stergioulas numerical model of a NS, to obtain the corresponding set of four α_n and hence answer at once the question about the type of the two Kerr-NUT sources constituting that concrete physically realistic numerical model. It is clear that the roots of equation (18) can give rise to the following four generic types of the two-soliton configurations for the NS models: type I configurations defined by four real-valued α 's that correspond to two subextreme Kerr-NUT

sources; type II configurations defined by two real α 's and a pair of complex conjugate α 's describing one subextreme and one hyperextreme Kerr-NUT constituents; type III configurations defined by two pairs of complex conjugate α 's determining a pair of identical hyperextreme Kerr-NUT constituents located above and below the equatorial plane; and lastly, type IV configurations defined by four pure imaginary α 's describing a pair of non-identical overlapping Kerr-NUT constituents located in the equatorial plane (see Fig. 1). In what follows it will be interesting to see whether matching of the Berti-Stergioulas numerical solutions for NSs to the MMR vacuum solution involves all four sectors of the two-soliton solution or only part of them.

At this point, we find it instructive to rectify one imprecise characteristic given by Pappas and Apostolatos in [20] to the paper [2], according to which the latter paper only presents an “algorithm” for the construction of exact solutions, requiring (in the pure vacuum case) resolution of equation (16) for some prescribed axis data. In reality, the “algorithm” mentioned by Pappas and Apostolatos is just Sibgatullin’s method itself, while the main output of the paper [2] is of course the $6N$ -parameter electrovac metric in which the quantities α_n , $n = \overline{1, 2N}$, are contained as *arbitrary* parameters. In [2] it was revealed for the first time that the analytical resolution of the algebraic equation for the axis data, which had been always considered a necessary step in Sibgatullin’s method but in practice had been restricting the method’s coverage mainly by the two-body systems (since already the generic three-body configuration leads to the algebraic equation of the sixth order), is not really needed, as the roots of that algebraic equation themselves can be introduced as arbitrary parameters into the multi-soliton solutions instead of certain parameters of the axis data. As a matter of fact, it is precisely the paper [2] that had converted the method initially considered as limited by various researchers (including Sibgatullin himself) into a powerful tool for the analytical study of the multi-black-hole configurations. In order to illustrate the above said, let us see how the MMR vacuum solution is contained in the extended double-Kerr (EDK) solution whose Ernst complex potential \mathcal{E} has the form [25, 33]

$$\begin{aligned}\mathcal{E} &= E_-/E_+, \\ E_{\mp} &= (\alpha_2 - \alpha_3)(\alpha_1 - \alpha_4)[X_1 r_1 - X_2 r_2 \mp (\alpha_1 - \alpha_2)][X_3 r_3 - X_4 r_4 \mp (\alpha_3 - \alpha_4)] \\ &\quad - (\alpha_1 - \alpha_2)(\alpha_3 - \alpha_4)[X_2 r_2 - X_3 r_3 \mp (\alpha_2 - \alpha_3)][X_1 r_1 - X_4 r_4 \mp (\alpha_1 - \alpha_4)], \\ r_n &= \sqrt{\rho^2 + (z - \alpha_n)^2}, \quad X_n = \frac{(\alpha_n - \bar{\beta}_1)(\alpha_n - \bar{\beta}_2)}{(\alpha_n - \beta_1)(\alpha_n - \beta_2)},\end{aligned}\tag{19}$$

where β_1 and β_2 are arbitrary complex constants, and α_n , $n = \overline{1, 4}$, can take arbitrary real values or form complex conjugate pairs. Without lack of generality, the parameters α_n can be assigned the order $\text{Re}(\alpha_1) \geq \text{Re}(\alpha_2) \geq \text{Re}(\alpha_3) \geq \text{Re}(\alpha_4)$, and it can be easily verified that on the upper part of the symmetry axis ($\rho = 0$, $z > \text{Re}(\alpha_1)$) the equation (16) corresponding to the EDK solution (19) rewrites as

$$\frac{2 \prod_{n=1}^4 (z - \alpha_n)}{\prod_{l=1}^2 (z - \beta_l)(z - \bar{\beta}_l)} = 0\tag{20}$$

for any combination of real and complex conjugate α 's, whence it follows immediately that α_n are indeed the roots of equation (16) from Sibgatullin’s method, although we did not have to find them by solving the latter equation.

It was already pointed out in [33] that the equatorially symmetric subfamily of the EDK solution is defined by the conditions

$$\begin{aligned}\alpha_1 + \alpha_4 &= \alpha_2 + \alpha_3 = 0, \\ X_1 X_4 &= X_2 X_3 = -1,\end{aligned}\tag{21}$$

so that formulas (19) and (21) combined together are equivalent to the MMR vacuum solution and, importantly, they permit us to give below a systematic description of all sectors of this solution depicted in Fig. 1.

(a) *Type I configurations in MMR solution.* When all α_n are real-valued, the corresponding constant quantities X_n , as can be readily seen from their definition in (19), satisfy the relations $X_n \bar{X}_n = 1$, hence being the unit complex constants. In this case, accounting for (21), the solution can be parametrized by the following four quantities,

$$\{\alpha_1, \alpha_2, X_1, X_2\},\tag{22}$$

the remaining parameters α_3 , α_4 , X_3 and X_4 being determined in terms of these as

$$\alpha_3 = -\alpha_2, \quad \alpha_4 = -\alpha_1, \quad X_3 = -1/X_2, \quad X_4 = -1/X_1.\tag{23}$$

Both Kerr-NUT constituents defined by this branch are subextreme.

(b) *Type II configurations in MMR solution.* In these configurations composed of one subextreme and one hyperextreme Kerr-NUT constituents, the constants α_1 and α_4 are real-valued, while α_2 and α_3 are complex conjugate pure imaginary parameters $\alpha_2 = \bar{\alpha}_3 = ip$, where p is an arbitrary real constant. Then it follows from (19) that

$$X_1 \bar{X}_1 = X_4 \bar{X}_4 = 1, \quad X_2 \bar{X}_3 = 1, \quad (24)$$

thus identifying X_1 and X_4 as unit complex constants, while X_2 and X_3 , with account of (21), are pure imaginary quantities. This sector of the MMR solution then can be parametrized by the quantities

$$\{\alpha_1, \alpha_2(=ip), X_1, X_2(=iq)\}, \quad (25)$$

where q is an arbitrary real constant, and the remaining parameters $\alpha_3, \alpha_4, X_3, X_4$ are related to the above α 's and X 's via the formulas (23), as in the previous case.

(c) *Type III configurations in MMR solution.* This case is characterized by two pairs of complex conjugate α 's, $\alpha_1 = \bar{\alpha}_2$ and $\alpha_3 = \bar{\alpha}_4$, describing two hyperextreme Kerr-NUT constituents separated by the equatorial plane, the corresponding X_n satisfying the relations

$$X_1 \bar{X}_2 = X_3 \bar{X}_4 = 1. \quad (26)$$

The additional conditions (21) of equatorial symmetry then imply that this sector of the MMR solution can be parametrized by only two arbitrary complex quantities

$$\{\alpha_1, X_1\}, \quad (27)$$

all the rest parameters being expressible in terms of α_1 and X_1 only:

$$\alpha_2 = -\alpha_3 = \bar{\alpha}_1, \quad \alpha_4 = -\alpha_1, \quad X_2 = 1/\bar{X}_1, \quad X_3 = -\bar{X}_1, \quad X_4 = -1/X_1. \quad (28)$$

(d) *Type IV configurations in MMR solution.* In this case, the two hyperextreme Kerr-NUT sources defined by four pure imaginary α 's lie entirely in the equatorial plane, and the constants X_n , as it follows from (19), are related by the equations

$$X_1 \bar{X}_4 = X_2 \bar{X}_3 = 1. \quad (29)$$

Moreover, conditions (21) reveal that all X_n are pure imaginary quantities, so that the two-soliton solution in this case can be parametrized by two α 's and two X 's:

$$\{\alpha_1(=ip_1), \alpha_2(=ip_2), X_1(=iq_1), X_2(=iq_2)\}, \quad (30)$$

where p_i and q_i , $i = 1, 2$, are arbitrary real parameters, the remaining constant quantities α_3, α_4, X_3 and X_4 having the form

$$\alpha_3 = -ip_2, \quad \alpha_4 = -ip_1, \quad X_3 = i/q_2, \quad X_4 = i/q_1. \quad (31)$$

Note that all types of the configurations (a)–(d) involve four arbitrary real parameters, which suggests that all sectors of the MMR solution might also be equally important from the physical point of view. The above analysis of the possible combinations of subextreme and hyperextreme Kerr-NUT objects described by the MMR solution complements and extends the analogous analysis performed by Pappas and Apostolatos directly in terms of the proper parameters of that solution, and we believe that our consideration permits one to see more clearly the interrelations between different sectors of the extended two-soliton solution. In particular, it is apparent that sector (a) of the MMR solution is equivalent to the equatorially symmetric (“parallel angular momentum”) subfamily pointed out by Oohara and Sato [34] within the usual double-Kerr solution of Kramer and Neugebauer [35] describing exclusively the subextreme Kerr-NUT constituents; therefore, since both Oohara and Sato, and earlier Kramer and Neugebauer themselves had indicated how the Kerr black-hole metric is obtainable from the subextreme double-Kerr solution, the affirmation made by Pappas and Apostolatos about the absence of the Kerr metric among the configurations of type I is erroneous. As a matter of fact, the Kerr black hole spacetime is definitely contained in the $b = 0$ subfamily of the type I MMR solution.

Turning now to the numerical models of NSs constructed and discussed by Berti and Stergioulas in [10], it should be observed that their matching to the MMR vacuum solution needs all four sectors of the latter solution. This can be seen already by analyzing the first two sequences of numerical solutions from table 1 of [10] constructed with EOS A. In our Tables I and II we give the values of the multipole moments defining those numerical solutions and

the corresponding values of the parameters α_1 and α_2 obtained by solving equation (18) that determine a respective sector (type) of the MMR solution. It is really surprising that only two numerical models can be matched to the type I MMR analytic solution involving two subextreme Kerr-NUT constituents, whereas seven models from Table I and all nine models from Table II require sectors with at least one hyperextreme Kerr-NUT constituent for matching to the MMR solution. In addition, we have checked that the entire supramassive sequence of ten models from table 1 of [10] matches to the type II MMR solution. The situation is quite similar in the case of the EOS AU numerical models of [10] too, whose type was investigated by Pappas and Apostolatos in [20] using the improved values of the multipoles M_2 and J_3 ,³ and for which only one of the total thirty models matches to the type I MMR solution. This clearly shows, on the one hand, that the subextreme double-Kerr solution is not appropriate for approximating a generic NS model and, on the other hand, that the hyperextreme Kerr-NUT objects are physically important and play a fundamental role in the description of the exterior field of NSs.

Taking into account the historical importance of the Hartle-Thorne approximate solution [4] for astrophysics, it would certainly be of interest to conclude this section by considering an exact analog of this well-known 3-parameter spacetime. Fortunately, such an exact analog can be easily identified thanks to the important fact established by Pappas and Apostolatos in [20] – the octupole rotational moment of the Hartle-Thorne solution is equal to zero. This means that the particular 3-parameter specialization of the MMR vacuum solution with $J_3 = 0$ represents the desired exact “Hartle-Thorne” spacetime. From (2) it follows that J_3 vanishes when $b = a(a^2 + 2k)/k$, so that the substitution of this value of b into the equations (4), (5) and (7) formally solves the problem of describing the desired exact solution in explicit form. Nonetheless, we still find it advantageous to rewrite the MMR solution in terms of the dimensionless multipole moments j, q, s related to M_0, J_1, M_2 and J_3 by the formulas

$$j = \frac{J_1}{M_0^2}, \quad q = \frac{M_2}{M_0^3}, \quad s = \frac{J_3}{M_0^4}, \quad (32)$$

in order to make the passage to the “Hartle-Thorne” subcase really trivial. The reparametrized MMR solution worked out with the help of the results of the paper [25] then can be presented (after additionally setting $M_0 = m$) in the following final form:

$$\begin{aligned} \mathcal{E} &= \frac{A - B}{A + B}, \quad f = \frac{A\bar{A} - B\bar{B}}{(A + B)(\bar{A} + \bar{B})}, \quad e^{2\gamma} = \frac{(j^2 + q)^2(A\bar{A} - B\bar{B})}{16\delta^2\kappa_+^4\kappa_-^4R_+R_-r_+r_-}, \\ \omega &= \frac{2m(s - jq)}{j^2 + q} - \frac{2\text{Im}[G(\bar{A} + \bar{B})]}{A\bar{A} - B\bar{B}}, \\ A &= \kappa_-^2(R_+ - r_-)(R_- - r_+) - \kappa_+^2(R_+ - r_+)(R_- - r_-), \\ B &= m\kappa_+\kappa_-[(\kappa_+ + \kappa_-)(r_+ - r_-) + (\kappa_+ - \kappa_-)(R_- - R_+)], \\ G &= -zB + m\kappa_+\kappa_-[\kappa_-(R_+r_+ - R_-r_-) + \kappa_+(R_-r_+ - R_+r_-) \\ &\quad - 2m\delta(R_+ + R_- - r_- - r_+)], \end{aligned} \quad (33)$$

where

$$\begin{aligned} R_{\pm} &= \frac{1 - Y_{\pm}}{1 + Y_{\pm}} \sqrt{\rho^2 + \left(z \pm \frac{m}{2}(\kappa_+ + \kappa_-)\right)^2}, \quad r_{\pm} = \frac{1 - y_{\pm}}{1 + y_{\pm}} \sqrt{\rho^2 + \left(z \pm \frac{m}{2}(\kappa_+ - \kappa_-)\right)^2}, \\ \kappa_{\pm} &= \sqrt{1 + j^2 + 2q - \left(j + \frac{jq - s}{j^2 + q}\right)^2} \pm 2\delta, \quad \delta = \sqrt{\left(\frac{q^2 + js}{j^2 + q}\right)^2 - \left(j + \frac{jq - s}{j^2 + q}\right)^2}, \end{aligned} \quad (34)$$

and

$$Y_{\pm} = \frac{\frac{i(jq - s)}{j^2 + q}[\pm(\kappa_+ + \kappa_-) - 2] - 2ij}{\frac{(jq - s)^2}{(j^2 + q)^2} \pm (\kappa_+ + \kappa_-) - \kappa_+\kappa_- - 1}, \quad y_{\pm} = \frac{\frac{i(jq - s)}{j^2 + q}[\pm(\kappa_+ - \kappa_-) - 2] - 2ij}{\frac{(jq - s)^2}{(j^2 + q)^2} \pm (\kappa_+ - \kappa_-) + \kappa_+\kappa_- - 1}. \quad (35)$$

The “Hartle-Thorne” exact solution is then obtainable from the above formulas (33)-(35) by just putting $s = 0$ in them. Note that the functions R_{\pm} and r_{\pm} in (34) are defined in a slightly different way than in (5).

³ In Tables I and II we have used the original data from the paper [10].

IV. DISCUSSION AND CONCLUSIONS

It seems very likely that the recent research on the numerical and analytical modeling of the geometry around NSs is able to produce a real breakthrough in our understanding of the important role that the hyperextreme sources play in general relativity. First of all, it is now clear that the inequality $j < 1$, defining the black-hole sector of the Kerr metric, in application to NSs may define a variety of configurations comprised of the subextreme and hyperextreme Kerr-NUT constituents. Indeed, although for all the numerical Berti-Stergioulas models of NSs, as for all astrophysical NSs in general, the inequality $j < 1$ between the total angular momentum and total mass holds, this inequality can be satisfied, within the framework of the two-soliton solution not only by the subextreme constituents, but by the hyperextreme constituents too. Let us illustrate this with the following simple calculation. Suppose, we have a system of N identical corotating hyperextreme Kerr sources characterized by the individual dimensionless angular momentum/mass ratio $j_0 > 1$ each. Then it is trivial to see that the corresponding ratio $j = J/M^2$ involving the total angular momentum J and total mass M of the system will be $j = j_0/N$, and hence for all $1 < j_0 < N$ we will have $j < 1$, so that this system formed exclusively by hyperextreme objects will be seen as a subextreme source if we formally apply to it the same criterion of subextremality as in the case of a single Kerr black hole. Moreover, it is obvious that in the type IV two-soliton configurations considered in the previous section, the hyperextreme sources lying in the equatorial plane can be counterrotating and therefore have in principle arbitrarily large individual spin parameters and at the same time give rise to the system's $j < 1$. Consequently, it would be plausible to suppose that the question of whether or not the sources of certain exact solutions describing the exterior geometry of physically realistic NS models are subextreme or hyperextreme has no much sense, especially in the context of the global NS models involving both the exterior and interior solutions where the regions with singularities do not show up at all; the only thing that really matters is the capacity of exact solutions to describe all types of sources.

Quite interestingly, the issue of the hyperextreme sources in the NS models turns out to be intimately related to the universal properties of NSs, in particular to a very attractive idea proposed by Pappas and Apostolatos [21] that the exterior gravitational field of a generic NS is determined by only four arbitrary multipole moments. The latter idea was corroborated by the explanation of possible mechanisms lying behind the situation when the internal degrees of freedom encoded in different EOS of NSs all lead to the exterior geometry described by just four multipoles, and it looks to us fairly feasible. It is quite logic to think that if a Kerr black hole [30] is described by an equatorially symmetric 2-parameter one-soliton solution, then the next-to-a-black-hole most compact and densest stellar object, a NS, must be described by the next equatorially symmetric soliton metric, the two-soliton one, possessing four arbitrary parameters. If this is the case, the entire multipole structure of NS models must be determined by only four arbitrary multipoles, thus giving rise to a sort of a “no-hair” theorem for NSs. In this respect it is worth noting that such a theorem, if correct, would cover both the subextreme and hyperextreme sources constituting the NS models, whereas the analogous “no-hair” theorem for uncharged black holes usually involves only the subextreme sector of the Kerr solution. However, if, in application to a Kerr black hole, we interpret the “no-hair” theorem as discriminating a spacetime geometry defined by only two parameters – mass and angular momentum – then it is not quite clear why we can not say for instance that “a Kerr naked singularity has no hair” too. Indeed, the multipole structure of the Kerr solution is concisely described by the formula [36]

$$M_n + iJ_n = m(ia)^n, \quad (36)$$

m being the source's mass and a its angular momentum per unit mass, and it is obvious that the above formula equally applies to the black-hole case ($m^2 \geq a^2$) characterized by the presence of an event horizon, and to the hyperextreme case ($m^2 < a^2$) characterized by the presence of a naked singularity. Therefore, Kerr's multipole structure is independent of whether the event horizon or a naked singularity is going to be formed during the gravitational collapse, so that in both cases the same mechanism must lie behind the structure's formation, the quantitative aspect of the mass-angular-momentum relation playing a secondary role in this process. Apparently, a key requirement this mechanism should meet is to be able to trigger off the *final* gravitational collapse.

It would probably be worth noting that during the late 70s and early 80s of the last century the solution generating techniques themselves had contributed a lot into drawing an artificial dividing line between subextreme and hyperextreme spacetimes, when the former spacetimes were generated with the aid of one ansatz and the latter with the aid of a different one. The appearance of Sibgatullin's integral method based on the general symmetry transformation for the Ernst equations has changed that situation drastically, as the extended multi-soliton solutions constructed with its help contain parameters corresponding to arbitrary multipole moments and hence describe in a unified manner both the subextreme and hyperextreme sources. For example, in the case of the Kerr solution, a starting point in Sibgatullin's method is the axis data

$$\mathcal{E}(0, z) = \frac{z - m - ia}{z + m - ia}, \quad (37)$$

where the real parameters m and a define two arbitrary multipoles $M_0 = m$ and $J_1 = ma$, so that at the output we will have

$$\mathcal{E}(\rho, z) = \frac{(\kappa - ia)r_+ + (\kappa + ia)r_- - 2m\kappa}{(\kappa - ia)r_+ + (\kappa + ia)r_- + 2m\kappa},$$

$$r_{\pm} = \sqrt{\rho^2 + (z \pm \kappa)^2}, \quad \kappa = \sqrt{m^2 - a^2}, \quad (38)$$

which is the Ernst potential of the Kerr spacetime that automatically describes a black hole or a hyperextreme object, depending on whether m^2 is greater or less than a^2 . For years, we have been calling for equal treatment of all types of solutions arising within the extended solitonic spacetimes, and we are really glad that the research of Berti and Stergioulas, and that of Pappas and Apostolatos, has finally produced a convincing evidence, confirmed by the present paper, that the exterior gravitational field of a generic NS can be appropriately described exclusively by the extended soliton solution due to a natural combination of its sub- and hyperextreme sectors.

Let us emphasize that we would be the first ones to admit that the vacuum MMR solution, well suited for analytical approximation of the geometry around NSs, might not be quite adequate for modeling the exterior fields of other, less compact, stellar objects for which a larger number of parameters representing arbitrary multipole moments could be needed in the corresponding exact solutions. In this regard, one can think about potential importance the equatorially symmetric configurations of the extended triple-Kerr [37, 38] and quadruple-Kerr [39] solutions might have for astrophysical applications in the future.

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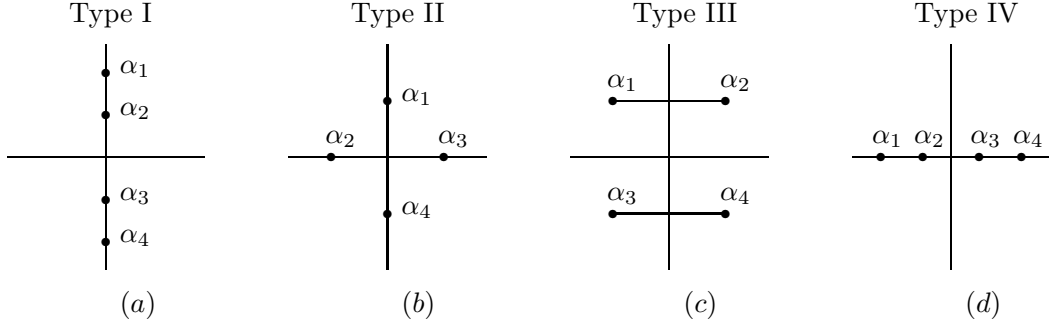


FIG. 1: Four different sectors of the MMR solution determined by combinations of the subextreme and hyperextreme Kerr-NUT constituents.

TABLE I: The multipole moments of the EOS A numerical models (normal sequence) from table 1 of the paper [10] and the corresponding types of the MMR solution matching those Berti-Stergioulas models.

M_0	J_1	M_2	J_3	α_1	α_2	Type
2.074	0.8121	-1.001	-0.727	1.862	0.111	I
2.081	1.327	-2.656	-3.158	1.386	0.484	I
2.089	1.704	-4.377	-6.694	0.877-0.614i	$\bar{\alpha}_1$	III
2.094	2.022	-6.173	-11.23	0.809-0.991i	$\bar{\alpha}_1$	III
2.102	2.307	-8.063	-16.77	0.732-1.272i	$\bar{\alpha}_1$	III
2.108	2.540	-9.806	-22.50	0.651-1.484i	$\bar{\alpha}_1$	III
2.114	2.729	-11.37	-28.10	0.568-1.650i	$\bar{\alpha}_1$	III
2.118	2.884	-12.74	-33.35	0.480-1.783i	$\bar{\alpha}_1$	III
2.118	2.925	-13.13	-34.88	0.454-1.821i	$\bar{\alpha}_1$	III

TABLE II: The multipole moments of the EOS A numerical models (second sequence) from table 1 of the paper [10] and the corresponding types of the MMR solution matching those Berti-Stergioulas models.

M_0	J_1	M_2	J_3	α_1	α_2	Type
2.453	0.9857	-0.623	-0.376	2.382	-0.156i	II
2.462	1.573	-1.657	-1.637	2.265	-0.265i	II
2.474	2.147	-3.237	-4.459	2.070	-0.377i	II
2.489	2.717	-5.434	-9.667	1.762	-0.513i	II
2.50	3.143	-7.529	-15.72	1.412	-0.652i	II
2.511	3.501	-9.619	-22.64	0.990	-0.837i	II
2.521	3.746	-11.31	-28.76	0.542	-1.028i	II
2.530	4.051	-13.52	-37.53	-1.430i	-0.334i	IV
2.537	4.247	-15.03	-44.00	-1.705i	-0.444i	IV